SELF-UNLINKED SIMPLE CLOSED CURVES

BY
DAVID W. HENDERSON(1)

1. **Discussion of results.** This paper is a sequel to [4] and all the definitions and notations of [4] will be assumed. In addition, the numbering of the theorems in the present paper has been made to follow the numbering of [4].

A simple closed curve J in a space M is said to be *self-unlinked*, if there exist a mapping $h: J \times [0,1] \to M$ such that

(a) $h \mid J \times \{0\}$ = inclusion of J in M,

(b) $h(J \times \{1\})$ = a point, and

(c) $h(J \times (0,1]) \subset M - J$.

In [4] we proved, as a partial answer to Question IV.1, that (IV.2) every self-unlinked tame simple closed curve (scc) in a 3-manifold bounds a disk. In this paper we investigate this question when we allow the scc's to be wild.

First we give some pertinent definitions, for which it will be assumed that everything is in a 3-manifold M. A complex is *wild* if it is not tame (see I. 11 of [4]). A *0-dimensional set is tame* if, for every $\varepsilon > 0$, it can be covered by the interiors of a collection of disjoint 3-cells each of diameter less than ε. A set X is *locally tame* at p if p has a closed neighborhood in X which is a tame complex in M. If X is not locally tame at p then p is a *wild point* of X. A set is called *nicely wild* if the union of its wild points is a tame 0-dimensional set.

For J an arc or scc we make the following definitions, the first of which is used in [1]. The *penetration index* $P(J,x)$ *of* J *at a point* $x \in J$ is the smallest cardinal number n such that there are arbitrarily small 2-spheres enclosing x and containing no more than n points of J. The *penetration index* $P(J)$ *of* J is the least upper bound of the cardinal numbers $P(J,x)$, for all $x \in J$. If J is nicely wild, then the *nice penetration index* $NP(J)$ *of* J is the smallest integer n such that, for every $\varepsilon > 0$, the set of wild points of J can be covered by the interiors of a collection of disjoint 3-cells each with diameter less than ε and such that the boundary of each 3-cell intersects J in no more than n points. (The union of members of this collection is called a *taming ε-set of J of index n*.)

CONJECTURE. *There is a nicely wild scc J such that* $NP(J) \neq P(J)$.

Received by the editors February 20, 1965.

(1) This paper consists of portions of the author's Ph.D. thesis written under the supervision of R. H. Bing and while the author was a National Science Foundation Graduate Fellow. The author wishes to thank the referee for his helpful suggestions.

The author expects such an example because he knows of a nicely wild scc J which has a point x such that $P(J,x)=3$; and, for any J, $NP(J)$ is even.

In the definition of nice penetration index we may require that the 3-cells are tame, because of the following:

THEOREM V.1. *Suppose every set of diameter less than ε in M lies in the interior of a convex 3-cell. (For instance, metrize M with the barycentric metric and let ε be less than 1.) **If** J is a nicely wild scc that is locally polyhedral* mod *its wild points, and if T is a taming ε-set of J of finite index, **then** there is a polyhedral taming ε-set T' of J with the same number of components as T, such that* $\mathrm{Bd}\,T' \cap J$ *has no more points than* $\mathrm{Bd}\,T \cap J$ *and J pierces* $\mathrm{Bd}\,T'$ *at each point of intersection.*

The principal results of this section are the following theorems.

In each *J is a self-unlinked, nicely wild scc in a 3-manifold M, and we further suppose that J is locally polyhedral* mod W *($W \equiv$ set of wild points of J).*

THEOREM V. 2. *J bounds an s-disk D which is locally polyhedral* mod W, *and* $|J| \cap |\mathrm{int}\,D| = \varnothing$.

THEOREM V.3. ***If** either*

(a) *$NP(J)=2$, or*

(b) *$NP(J)$ is finite and J has only finitely many wild points,*

***then** there is an s-disk D' and a sequence $\{T_i\}$ such that*

(a) *for each i, T_i is a taming $\frac{1}{2}i$-set of A of index $NP(J)$,*

(b) $|\mathrm{Bd}\,D'| = J$ *and* $[D'|D'^{-1}(|D'|-W), \mathrm{Bd}\,D'|D'^{-1}(J-W)]$ *is in rnp in $M-W$, and*

(c) *for each i, there is an s-disk D_i such that*

(i) $(D_i, \mathrm{Bd}\,D_i)$ *is in rnp in* $(M - \mathrm{int}\,T_i, J + \mathrm{Bd}\,T_i)$(2),

(ii) $|D_i| \supset |D_{i-1}|$, *and*

(iii) *D' equals the limit of the D_i's, as maps.*

THEOREM V.4. *If J bounds an s-disk D' satisfying the stated conclusion of Theorem* V.3, *then J bounds a nonsingular disk D.*

THEOREM V.5. *If $NP(J) \leqq 4$ and J has only finitely many wild points, then* $NP(J)=2$.

An immediate consequence of V.4, V.5 and the characterization of tame scc's by O. G. Harrold, H. C. Griffith, and E. E. Posey in [3] is the following:

THEOREM V.6. ***If** either*

(a) *$NP(J)=2$, or*

(b) *$NP(J) \leqq 4$ and J has only finitely many wild points, **then** J is tame.*

---

(2) $\mathrm{Bd}\,T_i + J$ is not a 2-manifold, but everything makes sense since $S(D_i) \subset \mathrm{Bd}\,T_i$.

If J is a scc on Alexander's Horned Sphere, S, which contains all the wild points of S, then

(a) J is a wild, nicely wild scc,

(b) $NP(J) = P(J) = 4$, and

(c) J bounds a disk.

In addition, by "tying the Fox'-Artin knot with a pointed ribbon" one can obtain a scc J such that

(a) J is a wild, nicely wild scc with one wild point,

(b) $NP(J) = P(J) = 6$, and

(c) J bounds a disk.

Finally, by tying a convergent sequence of knots in a scc, one obtains a scc J such that

(a) J is a nicely wild scc with one wild point,

(b) $NP(J) = P(J) = 2$,

(c) but J is wild.

2. **Proof of V.1.** Let C be a component of T. Theorem V.1 will follow if we produce a polyhedral 3-cell B such that $W \cap \text{int } C = W \cap \text{int } B$ (W= set of wild points of J), J pierces Bd B at each point of $J \cap \text{Bd } B$, diameter of $B < \varepsilon$, Bd $B \cap J$ has no more points than Bd $C \cap J$, and B does not intersect any other components of T.

Let δ be a positive number less than each of (ε – (diameter of C)), (1 / 3) (distance from C to $T - C$), and (1 / 3) (distance from Bd C to W). By the approximation theorems of [2] we may assume that Bd C is locally polyhedral mod $J \cap \text{Bd } C$.

Enclose each point p of Bd $C \cap J$ by a polyhedral 2-sphere S_p such that each S_p is so small that

(a) the diameter of S_p is less than δ,

(b) the S_p's are disjoint,

(c) $S_p \cap J$ is two points at each of which J pierces S_p,

(d) S_p is in general position with respect to Bd C, and

(e) there is a component K of Bd $C - \sum S_p$ which separates the (Bd $C \cap S_p$)'s on Bd C.

cl(K) (cl = closure) is a disk with holes and each component of Bd(cl(K)) is a scc on some S_p. For each p, only one scc of cl(K) $\cap S_p$ bounds a disk in Bd $C - K$ that intersects J. Therefore, since by hypothesis $C + \sum S_p$ is contained in the interior of a 3-cell, we may use linking arguments in E^3 to show that, for each p, all components but one of cl(K) $\cap S_p$ bounds a disk on $S_p - J$, and that the other one bounds a polyhedral disk on S_p that intersects J at most once. We can make these disks disjoint by pushing their interiors slightly to one side. Then K plus the above disks is a polyhedral 2-sphere S in a convex 3-cell of M. Let B be the 3-cell bounded by S.

Clearly Bd $B \cap J$ has no more points than Bd $C \cap J$, diameter of $B < \varepsilon$, and B does not intersect any other components of T. Let $w \in W \cap C$ and let λ be a general position arc from w to $M - (C + \sum S_p)$ which misses the S_p's. [This arc is possible since the 3-cells bounded by S_p have diameter less than (1/3) (distance from $\mathrm{Bd}\, C$ to W).] Then $\lambda \cap \mathrm{Bd}\, B = \lambda \cap K = \lambda \cap \mathrm{Bd}\, C$; thus, since $B + C$ is in the interior of a 3-cell and $w \in C$, $\lambda \cap \mathrm{Bd}\, B$ is an odd number of points and, therefore, $w \in B$. B is the desired 3-cell.

3. **Proof of V.2.** The proof of V.2 parallels the proof of III.4 of [4] and thus will only be sketched here.

$|J| - W$ is an infinite 1-dimensional polyhedral graph in M. Since J is self-unlinked we may assume that J is the boundary of an s-disk D and that D is polyhedral mod J. Consider $|J| - W$ as a subcomplex of some subdivision α of $M - W$.

Let Δ be the standard disk and let $\Omega \equiv D^{-1}(W)$.

Now go through the proof of III.4 replacing M by $M - W$, $|L|$ by $|J| - W, D$ by $D|(\Delta - \Omega)$, Δ by $\Delta - \Omega$, et cetera. Choose Δ' so that $\mathrm{Bd}\, \Delta' \cap \mathrm{Bd}\, \Delta = \Omega$.

4. **Proof of V.3.** We shall assume that M is so metrized that every set of diameter no more than 1 lies in a convex 3-cell (for example, the barycentric metric).

Let D be the disk promised by V.2 and (using II.2 of [4] in $M - W$) suppose that $(D|D^{-1}(|D| - W), \mathrm{Bd}\, D|D^{-1}(J - W))$ is in rnp in $M - W$.

Let $\varepsilon_1 = 1$, if $NP(J) = 2$; otherwise let ε_1 be a positive number less than 1 and so small that, if k is the number of points in W, then there is a positive integer $n \leqq NP(J) \times k$ such that

(4.1) no taming ε_1-set T of J of index $\leqq NP(J)$ has fewer than k components nor does $\mathrm{Bd}\, T \cap J$ have fewer than n points.

Let $\delta(\varepsilon)$ be a positive number less than $\varepsilon/3$ so small that

(4.2) if Δ' is a subdisk of Δ (the standard disk) and $\mathrm{diam}(D(\mathrm{Bd}\, \Delta')) < \delta(\varepsilon)$, then $\mathrm{diam}\, D(\Delta') < \varepsilon/3$.

Let $\Delta_1, \Delta_2, \cdots, \Delta_i, \cdots$ be an expanding sequence of proper subdisks of $\mathrm{int}\, \Delta$ so that $\{\mathrm{Bd}\, \Delta_i\}$ converges uniformly to $\mathrm{Bd}\, \Delta$.

Choose T_1 so that

(4.3) T_1 is a polyhedral (see VI.1) taming $\delta(\varepsilon_1)$-set of J of index $NP(J)$,

(4.4) all components of T_1 intersect W,

(4.5) if $NP(J) \neq 2$, T_1 has only k components and $\mathrm{Bd}\, T_1 \cap J$ has n points (see (4.1)), and

(4.6) $T_1 \subset M - D(\Delta_1)$.

We may suppose that $\mathrm{Bd}\, T_1$ and D are in general position so that

$$D^{-1}(\mathrm{Bd}\, T_1 \cap |D|)$$

is a finite collection of disjoint scc's and spanning arcs of Δ in $\Delta - \Delta_1$.

Let K be the component of $\Delta - D^{-1}(\mathrm{Bd}\, T_1 \cap |D|)$ containing Δ_1. The boundary of K is a finite collection of scc's in $D^{-1}(\mathrm{Bd}\, T_1 \cap |D|) + \mathrm{Bd}\, \Delta$. Let E_1 be the smallest disk in Δ containing K. (Note that $\mathrm{Bd}\, K \cap \mathrm{Bd}\, \Delta \subset \mathrm{Bd}\, E_1 \subset \mathrm{Bd}\, K$.)

If A is a member of $\mathfrak{A}$ (those scc's of $D^{-1}(\mathrm{Bd}\, T_1) \cap \mathrm{int}\, E_1$ which can be shrunk to a point in $\mathrm{Bd}\, T_1 - J$) and E_a is the disk that A bounds in E_1, then we can replace $D(E_a)$ by the singular disk which $D(A)$ bounds on $\mathrm{Bd}\, T_1 - J$. By pushing this disk slightly to one side of $\mathrm{Bd}\, T_1$ we can remove a component of $D^{-1}(\mathrm{Bd}\, T_1 \cap |D|)$. If we apply the above "disk-switching and pushing" only to outermost (in E_1) members of $\mathfrak{A}$ then no point of Δ will have its image changed more than once.

Thus, by applying the "disk-switching and pushing" to each outermost (in E) member of $\mathfrak{A}$ and then II.2 of [4] we obtain an s-disk D_1' such that

(4.7) $D_1' | \Delta_1 + (\Delta - \mathrm{int}\, E_1) + \mathrm{Bd}\, \Delta = D | \Delta_1 + (\Delta - \mathrm{int}\, E_1) + \mathrm{Bd}\, \Delta$,

(4.8) $D_1'^{-1}(\mathrm{Bd}\, T_1 \cap |D_1'|) \cap \mathrm{int}\, E_1$ is a finite collection of scc's whose images under D_1' cannot be shrunk to a point on $\mathrm{Bd}\, T_1 - J$, and

(4.9) $(D_1' - W, \mathrm{Bd}\, D_1' - W)$ is in rnp in $M - W$.

Let $\mathfrak{B}$ be the collection of all components (scc's) of $D_1'^{-1}(\mathrm{Bd}\, T_1 \cap |D_1'|) \cap \mathrm{int} E_1$. If $\mathfrak{B} \neq \varnothing$, let A be an innermost (in E_1) scc of $\mathfrak{B}$. A bounds a disk $E_a \subset E_1$ and $D_1'(E_a) \subset |D_1'| - \mathrm{int}\, T_1$, or $T_1 - J$. We shall treat these two cases separately.

If $D_1'(E_a) \subset T_1 - J$, then, since $D_1'(A)$ cannot be shrunk on $\mathrm{Bd}\, T_1 - J$, we can use the loop theorem to get a scc J_a such that J_a bounds a disk D_a in $T_1 - J$ but each of the two disks which J_a bounds on $\mathrm{Bd}\, T_1$ contain points of $J \cap \mathrm{Bd}\, T_1$. Thus D_a separates $J \cap C$, where C is the component of T_1 containing J_a. If we "cut" C apart along D_a (this cut could be accomplished by removing from C the interior of a regular neighborhood of D_a that misses J), we obtain a new taming δ-set T' of J.

If $NP(J) = 2$, then $J \cap \mathrm{Bd}\, C$ is two points and J intersects the boundary of each part of the "cut apart" C in only one point. But a scc that intersects a 2-sphere only once is contained wholly in one complementary domain or the other; therefore, $J \cap C$ is two points and C contains no points of W. This is a contradiction of (4.4).

If $NP(J) \neq 2$, then T' is a taming $\delta(\varepsilon_1)$-set of index $NP(J)$ and with $k+1$ components. But since W has only k points one of the components, C' say, of T' does not intersect W. But then $T' - C'$ is a taming $\delta(\varepsilon_1)$-set of index $NP(J)$ and with k components such that $\mathrm{Bd}(T' - C') \cap J$ has fewer points than $\mathrm{Bd}\, T \cap J$ which contradicts (4.1), (4.3), and (4.5).

Thus $D_1'(E_a)$ is not contained in $T_1 - J$.

If $D_1'(E_a) \subset |D_1'| - (\mathrm{int}\, T_1 + J)$, then by the loop theorem there is a real disk E_a such that $\mathrm{int}\, E_a$ is contained in $M - (T_1 + J)$. Also each of the disks E_a' and E_a'' which $\mathrm{Bd}\, E_a$ bounds on $\mathrm{Bd}\, T_1$ contains points of $J \cap \mathrm{Bd}\, T_1$. Because of (4.2) and (4.3), the diameter of E_a is less than $\varepsilon_1/3$. Thus one of $E_a + E_a'$ or $E_a + E_a''$, say $E_a + E_a'$, is a 2-sphere of diameter less than $2\varepsilon_1/3$ not containing C (the component of T_1 containing $\mathrm{Bd}\, E_a$) in its small complementary domain. Thus $E_a + E_a'$ lies in a convex 3-ball of M (see note at beginning of §4) and thus bounds a 3-cell B of diameter less than $2\varepsilon_1/3$.

$C + B$ is a 3-cell and $J \cap \mathrm{Bd}(C + B) = J \cap E''_a$ has fewer points than $J \cap \mathrm{Bd}\, C$. Thus, if $NP(J) = 2$, $J \cap \mathrm{Bd}(C + B)$ is one point and $C \subset C + B$ does not intersect W, which contradicts (4.4). If $NP(J) \neq 2$, then $\mathrm{Bd}(T_1 + B) \cap J$ has fewer points than $\mathrm{Bd}\, T_1 \cap J$ which contradicts (4.1), (4.2), (4.3), and (4.5).

Thus we conclude that $\mathfrak{B}$ is empty and that $D_1 = D'_1 | E_1$ is an s-disk satisfying (c) (i) of V.3, if D'_1 is substituted for D'. With the same substitution T_1 and D'_1 satisfy (a) and (b) of V.3.

We now repeat the above process letting ε_2 be a positive number less than ε_1 and $\frac{1}{2}$ and with the following substitutions: ε_2 for ε_1, D'_1 for D, D'_2 for D'_1, E_2 for E_1, $\Delta_2 + E_1$ for Δ_1, D_2 for D_1, and T_2 for T_1. We can choose T_2 to satisfy $T_2 \subset M - D'_1(\Delta_2 + E_1)$ since $D'^{-1}_1(W) = D^{-1}(W) \subset \mathrm{Bd}\, \Delta - E_1$. Thus, T_1, T_2, D_1, D_2, D'_2 satisfy (a), (b), and (c) (i) and (ii) of V.3 with D' replaced by D'_2.

We repeat the process at the ith stage after letting ε_i be a positive number less than ε_{i-1} and $1/2^{i-1}$ and then substituting ε_i for ε_1, D'_{i-1} for D, D'_i for D'_1, E_i for E_1, $\Delta_i + E_{i-1}$ for Δ_1, D_i for D_1, and T_i for T_1. Thus for each $i, T_1, T_2, \cdots, T_i$, $D_1, D_2, \cdots, D_i, D'_i$ satisfy (a), (b), and (c) (i) and (ii) of V.3 with D' replaced by D'_i.

By (4.7)

$$D'_i | \Delta_i + E_{i-1} + (\Delta - \mathrm{int}\, E_i) + \mathrm{Bd}\, \Delta = D_{i-1} | \Delta_i + E_{i-1} + (\Delta - \mathrm{int}\, E_i) + \mathrm{Bd}\Delta$$

and, since $E_i \supset \Delta_i + E_{i-1}$ and $\{\mathrm{Bd}\, \Delta_i\}$ converges to $\mathrm{Bd}\, \Delta$, every $p \in \mathrm{int}\, \Delta$ is in $\Delta_i + E_{i-1}$ for some i and thus $D'_j(p) = D'_{i-1}(p)$, for all $j \geqq i$. In addition, for each i, $D'_i | \mathrm{Bd}\, \Delta = D | \mathrm{Bd}\, \Delta$. Also the diameter of each component of $\Delta - E_i$ approaches zero as i approaches infinity and, for all p, the distance between $D_i(p)$ and $D'_{i+1}(p)$ is less than $\varepsilon_{i+1} < 1/2^i$. Thus $D' = \lim D'_i = \lim D_i$ is the s-disk desired for V.3.

5. **Proof of V.4.** Let D', $\{D_i\}$, $\{T_i\}$ be as given in the conclusion to V.3. Suppose α is a subdivision of $M - W$ so that $|D'| - W + \sum T_i$ is a subcomplex of $\alpha(M - W)$.

For $i = 1, 2, \cdots$, Theorem III.5 [applied to $(M - \mathrm{int}\, T_i, \mathrm{Bd}\, T_i + J)$] (see previous footnote) shows that there is an s-disk D'_i such that $(D'_i, \mathrm{Bd}\, D'_i)$ is a conservative δ_i-alteration of $(D_i, \mathrm{Bd}\, D_i)$, and $|D'_i|$ is related to $|D_i|$ as $|D'|$ is related to $|D^*|$ in the Addendum. We choose δ_i and $n(i)$ so that

$$(\delta_i\text{-neighborhood of } S(D'_i)) \subset \mathrm{st}[S(D'_i), \alpha^{n(i)}(M - \mathrm{int}\, T_i)] \subset M - J.$$

Thus, since $S(D'_i)$ contains, if anything, only crossing pinch points, $S(D'_i)$ is empty because $|\mathrm{int}\, D'_i| \subset M - \mathrm{int}\, T_i$. We also assume that each $|D'_i|$ is in general position with respect to each $\mathrm{Bd}\, T_j$.

For each i there is a positive integer $k(i)$ such that $|D_j| \supset |D'| \cap (M - \mathrm{int}\, T_i)$ for all $j \geqq k(i)$. Let $U_i = \mathrm{st}[S(D'_i), \alpha^{n(i)}(M - \mathrm{int}\, T_i)]$. Then, for all i and for all $j \geqq k(i)$

(a) $(|D'_j| - (U_i + \mathrm{int}\, T_i)) = (|D'| - (U_i + \mathrm{int}\, T_i))$, and

(b) $|D'_j| - \mathrm{int}\, T_i$ is related to $|D'| - \mathrm{int}\ T_i$ as $|D'|$ is related to $|D^*|$ in the Addendum.

There are only finitely many ways of putting things in U_i so that the Addendum is satisfied. Thus for some strictly increasing sequence of positive integers $\{n(1,i)\}$, $n(1,1) \geqq k(1)$, the pairs $[U_1, |D_{n(1,i)}| \cap U_1]$ are all pwl homeomorphic for $i = 1,2,3,\cdots$. Likewise there is a subsequence of $\{n(1,i)\}$ which we call $\{n(2,i)\}$ such that $n(2,i) \geqq k(n(1,1))$ and, for $i = 1,2,3,\cdots$, the pairs

$$[U_{n(1,1)}, |D'_{n(2,i)}| \cap U_{n(1,1)}]$$

are all pwl homeomorphic. In this way we get a sequence of sequences $\{n(1,i)\}$, $\{n(2,i)\}$, $\{n(3,i)\},\cdots$ such that $\{n(j,i)\}_{i=1}^{\infty}$ is a subsequence of $\{n(k,i)\}_{i=1}^{\infty}$ for all $k < j$, and, for each fixed k, the pairs $[U_{n(k,1)}, |D'_{n(k+1,i)}| \cap U_{n(k,1)}]$ are pwl homeomorphic for $i = 1,2,\cdots$.

Set $m(i) = n(i,1)$, for $i = 1,2,\cdots$. By moving things slightly in $\sum U_{m(i)}$ we can suppose that

$$|D'_{m(i)}| - \operatorname{int} T_{m(j)} = |D'_{m(k)}| - \operatorname{int} T_{m(j)}, \text{ for all } i,k > j.$$

The (nonsingular) s-disks $D'_{m(i)}$ are not nice enough because their limit might not be a disk. However, we shall choose certain subdisks and alter them to produce a nonsingular disk with boundary J.

Let E_1 be a sub-s-disk of $D'_{m(1)}$ such that

$$(5.2)_1 \qquad J + \operatorname{Bd} T_{m(1)} \text{ contains } |\operatorname{Bd} E_1|.$$

Let E_2 be a sub-s-disk of $D'_{m(2)}$ such that

$$(5.2)_2 \qquad J \cap |E_2| \subset |\operatorname{Bd} E_2| \subset J + \operatorname{Bd} T_{m(1)} \text{ and } |\operatorname{Bd} E_1| \subset |E_2|.$$

By induction, pick E_n to be a sub-s-disk of $D'_{m(n)}$ such that

$$(5.2)_n \qquad J \cap |E_n| \subset |\operatorname{Bd} E_n| \subset J + \operatorname{Bd} T_{m(n-1)} \text{ and } |\operatorname{Bd} E_{n-1}| \subset |E_n|.$$

PROPOSITION V.7. $J \subset \liminf\{|E_i|\}$.

Proof. By (5.2), $J \cap |E_i| \subset J \cap |E_{i+1}|$. Therefore, we need only show that every point of $J - W$ belongs to some $|E_i|$. Let q be any point of $J \cap |E_1|$ and let $p \in J - W$. For some positive integer r, $p \in M - T_{m(r)}$. Now suppose that $p \notin |E_{r+j}|$, for every $j \geqq 1$. Then, for each $j \geqq 1$, $T_{m(r+j-1)} \cap D'_{m(r+j)}$ separates p from q in $D'_{m(r+j)}$ and, because a disk is unicoherent, one component of $T_{m(r+j-1)} \cap D'_{m(r+j)}$ separates p from q. But $(p+q) \not\subset T_{m(r+j-1)}$ and each component of $T_{m(r+j-1)}$ has diameter less than $1/2^{m(r+j-1)}$. We conclude that, for every ε, there is a subset R of J which is of diameter less than ε and which is within ε of W, such that R separates p from q. But, since neither p nor q belong to W, some point of W must separate p from q in J. This is a contradiction since no scc is separated by a single point. This proves V.7.

PROPOSITION V.8. *For every positive integer r, there is a positive integer $s(r)$, such that, for all $i,j \geqq s(r)$,*

$$|E_j| - \operatorname{int} T_{m(r)} = |E_i| - \operatorname{int} T_{m(r)}.$$

Proof. $D'_{m(r+1)} - \operatorname{int} T_{m(r)}$ has finitely many components and if, for some i, $|E_i|$ intersects one of these components, then it contains the whole component. For each component C of $(D'_{m(r+1)} - \operatorname{int} T_{m(r)})$, let $n(C)$ be the least integer such that $C \subset |E_{n(C)}|$ and set $n(C) = 0$ if C intersects no $|E_i|$. The s desired by V.8 is the maximum of the $n(C)$'s over all components C of $D'_{m(r+1)} - \operatorname{int} T_{m(r)}$.

Define $s^n(r) = s(s^{n-1}(r))$.

We now change the E_i's into an expanding sequence of disks in a countable number of steps.

Step 1. Let F_1 be the singular s-disk gotten by removing from $E_{s(1)}$ the interior of $\operatorname{Bd} E_1$ in $E_{s(1)}$ (see (5.2)) and replacing it by E_1. Formally, let Δ' be the subdisk of Δ bounded by $E_{s(1)}^{-1}(\operatorname{Bd} E_1)$; and let f be a homeomorphism of Δ onto Δ' such that

$$(E_{s(1)}|\operatorname{Bd}\Delta') \circ (f|\operatorname{Bd}\Delta) = E_1|\operatorname{Bd}\Delta.$$

Then F_1 equals $E_{s(1)}$ on $\Delta - \operatorname{int}\Delta'$ and $E_1 \circ f^{-1}$ on Δ'. The singularities $S(F_1)$ are contained in $M - T_{m(1)}$. Let $\delta_1 = \frac{1}{2}$(distance from $S(F_1)$ to $T_{m(1)}$) and apply IV.3 of [4] to get a nonsingular s-disk F'_1 which is a conservative δ_1-alteration of F_1 such that $\operatorname{Bd} F'_1 = \operatorname{Bd} F_1 = \operatorname{Bd} E_{s(1)}$. Note that $F'_1 \subset M - T'_{m(s(1))}$.

Step 2. Let F_2 be the singular disk gotten by removing from $E_{s^2(1)}$ the interior of $\operatorname{Bd} E_{s(1)}$ in $E_{s^2(1)}$ and replacing it by F'_1. Since

$$|E_{s(1)}| - \operatorname{int} T_{m(1)} = |E_{s^2(1)}| - \operatorname{int} T_{m(1)}, \qquad |F_2| - |F'_1| \subset \operatorname{int} T_{m(1)}.$$

Thus, because

$$F'_1 \subset M - T_{m(s(1))}, \qquad S(F_2) \subset \operatorname{int} T_{m(1)} - T_{m(s(1))}.$$

Let $\delta_2 = \frac{1}{2}$(distance from $S(F_2)$ to $T_{m(s(1))}$) and apply IV.3 of [4] to get a nonsingular s-disk F'_2 which is a conservative δ_2-alteration of F_2. F'_2 has the following properties:

(5.3)$_2$ $\qquad E_1$ is a sub-s-disk of F'_2.

(5.4)$_2$ $\qquad \operatorname{Bd} F'_2 = \operatorname{Bd} F_2 = \operatorname{Bd} E_{s^2(1)}$.

(5.5)$_2$ $\qquad F'_2 \subset M - T_{m(s^2(1))}$.

(5.6)$_2$ $\qquad F'_1 - T_{m(1)} = F'_2 - T_{m(1)}$.

Step n $(n = 3, 4, \cdots)$. Let F_n be the singular s-disk gotten by removing from $E_{s^n(1)}$ the interior of $\operatorname{Bd} E_{s^{n-1}(1)}$ in $E_{s^n(1)}$ and replacing it by F'_{n-1} (see (5.2) and (5.4)$_{n-1}$). By V.8, $|F_n| - |F_{n-1}| \subset \operatorname{int} T_{m(s^{n-2}(1))}$. Thus, by (5.5)$_{n-1}$,

$$S(F_n) \subset \operatorname{int} T_{m(s^{n-2}(1))} - T_{m(s^{n-1}(1))}.$$

Let $\delta_n = \frac{1}{2}$(distance from $S(F_n)$ to $T_{m(s^{n-1}(1))}$) and apply IV.3 of [4] to get a nonsingular s-disk F' which is a conservative δ_n-alteration of F_n. F'_n has the following properties.

$(5.3)_n$ $\qquad F'_{n-2}$ is a sub-s-disk of F'_n.

$(5.4)_n$ $\qquad \operatorname{Bd} F' = \operatorname{Bd} E_{s^n(1)}$.

$(5.5)_n$ $\qquad F'_n \subset M - T_{m(s^n(1))}$.

$(5.6)_n$ $\qquad F'_{n-1} - T_{m(s^{n-1}(1))} = F'_n - T_{m(s^{n-1}(1))}$.

Define $E_1 \equiv F'_0$.

We now use the F'_i's to construct a nonsingular s-disk D whose boundary is J.

PROPOSITION V.9. *For all $m \geqq n \geqq 2$ and for all onto homeomorphisms $g: \Delta \to\to \Delta$, there is an onto homeomorphism $h^m_n(g): \Delta \to\to \Delta$, such that*

$$(F'_m \circ h^m_n(g)) \,|\, (F'_n \circ g)^{-1} \,|F'_{n-2}| = (F'_n \circ g) \,|\, (F'_n \circ g)^{-1} \,|F'_{n-2}|.$$

Proof. There is essentially only one way of extending a disk Δ' to a larger disk Δ when $\Delta' \cap \operatorname{Bd}\Delta$ is given. [That is to say, given $\Delta' \subset \Delta_1$ and $\Delta' \subset \Delta_2$ such that $\Delta' \cap \operatorname{Bd}\Delta_1 = \Delta' \cap \operatorname{Bd}\Delta_2$, there is a homeomorphism of Δ_1 onto Δ_2 fixed on Δ'.] From (5.2) and (5.4) we conclude that, for $m \geqq n$, $F'^{-1}_m(|\operatorname{Bd} F'_{n-2}|) \cap \operatorname{Bd}\Delta = F'^{-1}_m(|\operatorname{Bd} F'_{n-2}| \cap J)$. Proposition V.9 now follows.

Using V.9, define

$$F''_0 \equiv F'_0 = E_1,$$

$$F''_1 \equiv F'_1,$$

$$F''_2 \equiv F'_2,$$

and, for $n = 3, 4, 5, \cdots$,

$$F''_n = F'_n \circ h^n_{n-1}(F'^{-1}_{n-1} \circ F''_{n-1}).$$

The reader can check that F''_n, $n = 2, 3, \cdots$, satisfies $(5.3)_n$–$(5.6)_n$ with all primes (′) replaced by double-primes (″). In addition, if we define $\Delta_i = F''^{-1}_{i+2}(|F''_i|)$,

$(5.7)_n$ $\qquad$ for all $m \geqq n + 2 \geqq 4$, $\quad F''_m | \Delta_n = F''_{n+2} | \Delta_n$.

This follows from V.9.

Define $D | \Delta_i \equiv F''_{i+2} | \Delta_i$. By (5.7), D is a 1-1, continuous map of $\sum_{i=1}^{\infty} \Delta_i$ into M. Since each component of T_i is of diameter less than $1/2^i$, (5.5) and (5.6) show that D can be extended to a 1-1, continuous map (and thus, an embedding) of Δ into M. It follows from V.7 that $J \subset D(\Delta)$ and from (5.2) and (5.4) that $J = |\operatorname{Bd} D|$.

This completes the proof of V.6.

6. **Proof of V.5.** Let D be the nonsingular s-disk promised by Theorem V.4 and let ε_1, $\delta(\varepsilon_1')$, and T_1 be as in (4.1)–(4.5) with the additional requirement that T_1 be a γ-set, where γ is less than $\frac{1}{2}\delta(\varepsilon_1)$ and so small that if p and q are points of J within γ of each other then one of the components of $J-(p+q)$ has diameter less than $\frac{1}{2}\delta(\varepsilon_1)$. Assume that D is polyhedral mod W and that D and $\operatorname{Bd} T_1$ are in general position.

Let C be a component of T_1 such that $\operatorname{Bd} C \cap J$ has four points. Call these four points p_1, p_2, p_3, and p_4. $D^{-1}(\operatorname{Bd} C \cap |D|)$ is a finite collection of scc's and spanning arcs in Δ. Since the only possible end points for $D^{-1}(\operatorname{Bd} C \cap |D|)$ are $D^{-1}(\operatorname{Bd} C \cap J)$, $D^{-1}(\operatorname{Bd} C \cap |D|)$ has two spanning arcs which are situated as in Figure 1 or Figure 2. Note that $W \cap C$ is one point, which we call w; and

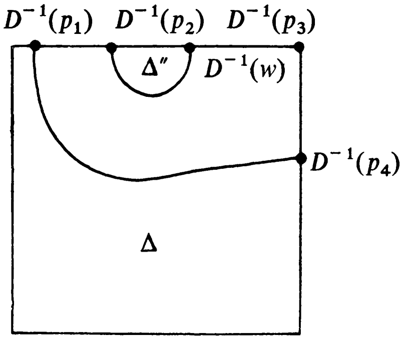

FIGURE 1

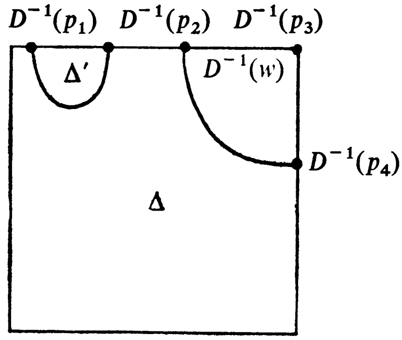

FIGURE 2

$D^{-1}(w)$ is between $D^{-1}(p_1)$ and $D^{-1}(p_2)$, or $D^{-1}(p_3)$ and $D^{-1}(p_4)$. We suppose the latter.

We shall look at the two cases depicted by Figure 1 and Figure 2.

Case depicted by Figure 1. The image of a neighborhood of $\operatorname{Bd}\Delta''$ is in $M - \operatorname{int} T_1$, and $(D|\Delta'')^{-1}(\operatorname{Bd} C \cap |(D|\Delta'')|)$ is a finite collection of scc's. By using arguments almost identical to those employed in the proof of V.3, we can remove the scc's that bound disks on $\text{Bd}\, C - J$ and then show that no more are left. Thus we suppose that D has been altered so that

$$(6.1)\qquad \begin{gathered} D \big| \Delta'' \text{ is nonsingular, } \operatorname{int}(D \big| \Delta'') \subset M - T_i, \text{ and} \\ \text{Bd}\,(D \big| \Delta'') \subset \text{Bd}\, C + (\text{small part of } J \text{ between } p_2 \text{ and } p_3). \end{gathered}$$

By our choice of γ, the diameter of $\big|\text{Bd}(D \big| \Delta'')\big|$ is less than $\delta(\varepsilon_1)$ and thus the diameter of $\big|(D \big| \Delta'')\big|$ is less than $\varepsilon_1/3$. (See (4.2).) By thickening up $\big|(D \big| \Delta'')\big|$ we obtain a new taming ε_1-set T' of J of index $\leqq 4$ and such that $J \cap \text{Bd}\, T'$ has two fewer points than $J \cap \text{Bd}\, T_1$. This contradicts (4.1) and (4.5).

Case depicted by Figure 2. By repetition of previous arguments we can alter D on int Δ' to remove the components of intersection with $\text{Bd}\, C$, and thus obtain a nonsingular s-disk D', such that $\big|\operatorname{int} D'\big| \subset \operatorname{int} C$ and $\text{Bd}\, D' \subset \text{Bd}\, C +$ (component of $J \cap C$ between p_1 and p_2). By splitting C apart along $\big|D'\big|$ we can obtain a new taming $\delta(\varepsilon_1)$-set T' of J of index $\leqq 4$ and such that $J \cap \text{Bd}\, T'$ has two fewer points than $J \cap \text{Bd}\, T_1$. This again is a contradiction.

Thus V.5 is proven.

REFERENCES

1. B. J. Ball, *Penetration indices and applications*, Topology of 3-manifolds and related topics, Prentice-Hall, Englewood Cliffs, 1962.

2. R. H. Bing, *Approximating surfaces with polyhedral ones*, Ann. of Math. **61** (1957), 456–483.

3. O. G. Harrold, H. C. Griffith, and E. E. Posey, *A characterization of tame curves in* 3-*space*, Trans. Amer. Math. Soc. **79** (1955), 12–35.

4. D. W. Henderson, *Extensions of Dehn's Lemma and the Loop Theorem*, Trans. Amer. Math. Soc. **120** (1965), 448–469.

UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN
INSTITUTE FOR ADVANCED STUDY,
PRINCETON, NEW JERSEY